On Six New Relations Between the Elastic Constants of Crystals

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It is shown that in those cases where Laval's scheme of 45 elastic constants is applicable, there are six relations that must be satisfied by the elastic constants. These relations follow from an analysis of the body-couples and of the strain-energy function. As a consequence the number of independent elastic constants of a triclinic crystal is reduced from 45 to 39; reductions also take place in the other crystal classes. The new elastic matrices are given for each of the eleven Laue-groups and for isotropic solids. There is also a discussion on the various assumptions that have been made by different authors regarding the rotational invariance of the strain energy.

1. Introduction

Until 1951 it was accepted that the elastic properties of the most general (least symmetrical) crystal could be described by means of 21 elastic constants. The most comprehensive of the early contributions in this field is to be found in Voigt's *Lehrbuch der Kristallphysik* (1910, 1928). Born *et al.* (1954) adapted their crystal dynamics to the scheme of the 21 elastic constants of Voigt. But in recent years there have been several theoretical publications, mainly by Laval (1952, 1957), LeCorre (1953, 1954, 1955, 1958), Viswanathan (1954, 1955), Raman (1955), and Joel & Wooster (1957), suggesting a more general approach which involves a greater number of elastic constants— 45 instead of 21 in the most general case, with corresponding increases in each of the crystal classes (LeCorre, 1953). There are also some accounts of experimental work confirming the need for an increase in the number of elastic constants for quartz (Zubov & Firsova, 1956) and ammonium dihydrogen phosphate (LeCorre, 1954a; Joel & Wooster, 1960).

In this paper the subject is discussed from the macroscopic point of view. An analysis of the body couples and of the strain-energy function shows that in those cases where the new theory might be applicable, some relations have to be satisfied by the elastic constants. These relations reduce the number of independent constants from 45 to 39. A brief account has been published already (Joel & Wooster, 1958).

2. Strains, stresses, rotations and body-couples

The strain tensor $\varepsilon_{\alpha\beta}$ is defined by

$$\varepsilon_{\alpha\beta} = (\partial u_\alpha / \partial x_\beta) \qquad (\alpha, \beta = 1, 2, 3), \qquad (1)$$

where $\mathbf{u}(u_1, u_2, u_3)$ is the displacement at the point $\mathbf{x}(x_1, x_2, x_3)$.

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It will be convenient, for the developments that will follow, to introduce the notation:

$$\varepsilon_{11} = \varepsilon_1, \quad \tfrac{1}{2}(\varepsilon_{32} + \varepsilon_{23}) = \varphi_1, \quad \tfrac{1}{2}(\varepsilon_{32} - \varepsilon_{23}) = \theta_1,$$
$$\varepsilon_{22} = \varepsilon_2, \quad \tfrac{1}{2}(\varepsilon_{13} + \varepsilon_{31}) = \varphi_2, \quad \tfrac{1}{2}(\varepsilon_{13} - \varepsilon_{31}) = \theta_2,$$
$$\varepsilon_{33} = \varepsilon_3, \quad \tfrac{1}{2}(\varepsilon_{21} + \varepsilon_{12}) = \varphi_3, \quad \tfrac{1}{2}(\varepsilon_{21} - \varepsilon_{12}) = \theta_3. \qquad (2)$$

The symmetric and antisymmetric parts of the strain tensor (pure deformation and pure rotation respectively) now take the forms:

$$\begin{vmatrix} \varepsilon_1 & \varphi_3 & \varphi_2 \\ \varphi_3 & \varepsilon_2 & \varphi_1 \\ \varphi_2 & \varphi_1 & \varepsilon_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{vmatrix}. \qquad (3)$$

ε_α is the longitudinal strain, or extension per unit length, along the axis X_α; $2\varphi_\alpha$ and θ_α are respectively the total shear deformation and the rotation around an axis parallel to X_α.

Until the work of Laval (1952) it was assumed that the rotations θ_α could be neglected. Such rotations in the crystal were considered irrelevant in relation to the elastic energy. Assumptions used in static elasticity were being applied to dynamic elasticity without further qualification. But when the strains are functions of $\mathbf{x}$ and t, then the rotations θ_α, the angular velocities $\dot{\theta}_\alpha$ and the angular accelerations $\ddot{\theta}_\alpha$ in every element of volume are also functions of $\mathbf{x}$ and t, and so are the stresses. For elastic waves they are all periodic functions of $\mathbf{x}$ and t with the same time-frequency ν. For such strains the rotations cannot be neglected, and therefore in the analysis that follows all nine components of the strain tensor are taken into consideration.

The stress tensor will be designated by $\sigma_{\alpha\beta}(\alpha, \beta = 1, 2, 3)$. $\sigma_{\alpha\beta}$ is the component of stress in the direction of the axis X_α acting on a surface perpendicular to X_β.

In every element of volume, the shear stresses $(\alpha \neq \beta)$ exert couples. Their moment per unit volume will be designated by $\mathbf{M}(M_1, M_2, M_3)$, so that:

$$M_1 = \sigma_{32} - \sigma_{23}$$
$$M_2 = \sigma_{13} - \sigma_{31}$$
$$M_3 = \sigma_{21} - \sigma_{12} . \tag{4}$$

Contrary to the usual treatments of the problem we shall now assume that body-couples may exist. The moment per unit volume exerted by these body-couples will be designated by $G(G_1, G_2, G_3)$, so that $G_\alpha dv$ is the moment acting on the element of volume dv around an axis parallel to X_α. If body-couples are neglected, the rotational equilibrium conditions require that the moments M_α should vanish. In this case equations (4) give:

$$\sigma_{\alpha\beta} - \sigma_{\beta\alpha} = 0; \tag{5}$$

that is, the stress tensor is symmetrical.

But when the body-couples are not neglected, then equation (5) takes a different form. Rotational equilibrium requires:

$$\mathbf{M} + \mathbf{G} = 0 \tag{6}$$

from which it follows that

$$\sigma_{32} - \sigma_{23} + G_1 = 0$$
$$\sigma_{13} - \sigma_{31} + G_2 = 0$$
$$\sigma_{21} - \sigma_{12} + G_3 = 0 . \tag{7}$$

and the stress tensor is not necessarily symmetrical. With this notation Hooke's law takes the form:

$$\sigma_{\alpha\beta} = \sum_{\gamma\delta} c_{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} \quad (\alpha, \beta, \gamma, \delta = 1, 2, 3) . \tag{8}$$

Using Laval's abridged notation this reduces to

$$\sigma_i = \sum_j c_{ij} \varepsilon_j \quad (i, j = 1, 2, \ldots, 9) \tag{9}$$

and the condition

$$c_{ij} = c_{ji}$$

must apply.* The relation between the four-index and the two-index notation proposed by Laval is given in the following table.

$\alpha\beta, \gamma\delta$	11	22	33	23	31	12	32	13	21	
$i, \quad j$	1	2	3	4	5	6	7	8	9 .	(10)

By combining equations (7) with Hooke's law (equations (8) and (9)) it is possible to express $\mathbf{G}$, the moment per unit volume exerted by the body-couples, in terms of the components of strain. Using the abridged notation it follows from equations (7) that

$$G_\beta = \sigma_{\beta+3} - \sigma_{\beta+6} \quad (\beta = 1, 2, 3) . \tag{11}$$

Introducing Hooke's law into this equation gives:

$$G_\beta = \sum_{i=1}^{9} (c_{\beta+3, i} - c_{\beta+6, i}) \varepsilon_i . \tag{12}$$

* Makinson (1955) has suggested that under certain circumstances it might be possible that $c_{ij} \neq c_{ji}$.

In order to show explicitly the deformations and the rotations, the relations (2) will be used. It follows from formulae (2):

$$\varepsilon_4 = \varepsilon_{23} = \varphi_1 - \theta_1$$
$$\varepsilon_5 = \varepsilon_{31} = \varphi_2 - \theta_2$$
$$\varepsilon_6 = \varepsilon_{12} = \varphi_3 - \theta_3$$
$$\varepsilon_7 = \varepsilon_{32} = \varphi_1 + \theta_1$$
$$\varepsilon_8 = \varepsilon_{13} = \varphi_2 + \theta_2$$
$$\varepsilon_9 = \varepsilon_{21} = \varphi_3 + \theta_3 . \tag{13}$$

If the substitutions (13) are introduced into equation (12), then the moments G_β become a function of the nine variables: $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varphi_1, \varphi_2, \varphi_3, \theta_1, \theta_2$ and θ_3. The terms can be grouped together in three parts: one that contains only the longitudinal strains ε_α; one with the shear deformations φ_α; and one in terms of the rotations θ_α. The expression for G_β becomes (all summations over α, from 1 to 3):

$$G_\beta = \sum (c_{\alpha, \beta+3} - c_{\alpha, \beta+6}) \varepsilon_\alpha$$
$$+ \sum (c_{\alpha+3, \beta+3} - c_{\alpha+3, \beta+6} + c_{\alpha+6, \beta+3} - c_{\alpha+6, \beta+6}) \varphi_\alpha$$
$$+ \sum (-c_{\alpha+3, \beta+3} + c_{\alpha+3, \beta+6} + c_{\alpha+6, \beta+3} - c_{\alpha+6, \beta+6}) \theta_\alpha . \tag{14}$$

The form of equation (14) suggests an analysis of the conditions under which the body-couples may exist; this will be given in the following section. But first it may be of interest to give an example of a qualitative model illustrating the concept of body-couples. Let us suppose that the crystal is composed of electric dipoles all in parallel orientation. If an external electric field is applied in a direction inclined to the electric axis of each dipole then a mechanical couple will be exerted on the crystal. The magnitude of this couple is proportional to the volume of the crystal, provided that the electric moments of the small dipoles are not affected by the applied field. Such a couple is properly described as a body-couple. The body-couples considered here are not, however, generated by an external field. We may assume that in certain crystals of sufficiently low symmetry the application of particular stresses gives rise to displacements of charge which produce an internal electric field. If the direction of this field does not coincide with the direction of the axis of the electric dipole in each unit cell, then a body-couple must be produced in the same way as described above. If the material has an isotropic distribution of polarizable atoms or if the symmetry is sufficiently high, then the directions of the internal field and of each dipole will coincide. However, if the atomic distribution is less symmetric there is no reason why there should not be a small angle between the internal field and the axis of the induced dipole. Similarly, if the effect of applying stresses were to induce magnetic dipoles, the internal magnetic field due to them might not coincide with the axis of the dipoles and again body couples would be produced.

3. The six new relations between the elastic constants

Whatever the mechanism responsible for the appearance of the body-couples, it seems reasonable to expect that, if they can exist at all, these strain-induced couples can only arise in elements of volume that are subject to deformation. It is in fact very difficult to conceive any strain-induced moment arising in an undeformed (purely rotated) element of volume, in which no structural change has occurred. This hypothesis can be formulated as follows:

Given a crystal subject to mechanical strains only, a necessary condition for the appearance of body-couples in any element of volume is that a deformation should be present in this element of volume. (It is not stated that this condition is sufficient!)

This hypothesis will now be applied to equation (14). If in an element of volume there is no deformation present, that is, $\varepsilon_\alpha = \varphi_\alpha = 0$ the expression for G_β is reduced to:

$$G_\beta = \sum_{\alpha=1}^{3} (-c_{\alpha+3, \beta+3} + c_{\alpha+3, \beta+6} + c_{\alpha+6, \beta+3} - c_{\alpha+6, \beta+6}) \theta_\alpha .$$

(15)

But, according to the hypothesis stated above, G_β should in this case be equal to zero. This requires that:

$$c_{\alpha+3, \beta+3} + c_{\alpha+6, \beta+6} = c_{\alpha+3, \beta+6} + c_{\alpha+6, \beta+3}$$
$$(\alpha, \beta = 1, 2, 3) .$$

(16)

Equation (16) contains six independent relations. They are:

$$2c_{47} = c_{44} + c_{77}$$
$$2c_{58} = c_{55} + c_{88}$$
$$2c_{69} = c_{66} + c_{99}$$
$$c_{45} + c_{78} = c_{48} + c_{57}$$
$$c_{56} + c_{89} = c_{59} + c_{68}$$
$$c_{46} + c_{79} = c_{49} + c_{67} .$$

(17)

The six relations (17) reduce by six the number of independent elastic constants of a triclinic crystal (Laue-group $\bar{1}$), leaving it at 39. In the other Laue-groups there is also a reduction as compared with the tables given by LeCorre (1953). The new elastic matrices for the 11 Laue-groups and for isotropic solids are given in Appendix 1.

The following table gives the number of independent

Laue-group	V	LL	JW
$\bar{1}$	21	45	39
$2/m$	13	25	21
mmm	9	15	12
$4/m$	7	13	11
$4/mmm$	6	9	7
$\bar{3}$	7	15	13
$3m$	6	10	8
$6/m$	5	11	9
$6/mmm$	5	8	6
$m3$	3	5	4
$m3m$	3	4	3
Isotropic	2	3	2

components for each of those 12 matrices according to the theory of Voigt (V), to that of Laval, LeCorre, Viswanathan & Raman (LL), and the one suggested in the present paper (JW).

It will be noticed that there is no difference between the new elastic constants and those of Voigt for crystals of the Laue-group $m3m$ and for isotropic solids: both the present formulation and that of Voigt require $c_{44} = c_{47}$ and $G_\alpha = 0$; the formulation of Laval, on the other hand, permits $c_{44} \neq c_{47}$ and $G_\alpha = 2(c_{47} - c_{44})\theta_\alpha$. But it seems hardly possible that there could be strain-induced couples acting in an isotropic solid.

As to the significance of the six new relations between the elastic constants, it is interesting to compare them with their analogues in the Voigt theory.

If the elastic constants are written in the four-index notation, with their full subscripts $\alpha\beta\gamma\delta$, equation (16) becomes:

$$c_{\alpha\beta\gamma\delta} + c_{\beta\alpha\delta\gamma} = c_{\alpha\beta\delta\gamma} + c_{\beta\alpha\gamma\delta} .$$

(18)

The analogue of (18) in the Voigt theory is:

$$c_{\alpha\beta\gamma\delta} = c_{\beta\alpha\delta\gamma} = c_{\alpha\beta\delta\gamma} = c_{\beta\alpha\gamma\delta} .$$

(19)

Relations (19) hold if the coefficients of ε_α and φ_α in equation (14) are also put equal to zero, that is, in the absence of body-couples.

4. Relations between body-couples and deformations

If the condition (16) is introduced into equation (14), the latter becomes:

$$G_\beta = \sum_\alpha (c_{\alpha, \beta+3} - c_{\alpha, \beta+6})\varepsilon_\alpha + 2\sum_\alpha (c_{\alpha+3, \beta+3} - c_{\alpha+3, \beta+6})\varphi_\alpha ,$$

(20)

which gives the moments per unit volume exerted by the body-couples in terms of the deformations.

If equation (20) is written:

$$G_\beta = b_{\beta 1}\varepsilon_1 + b_{\beta 2}\varepsilon_2 + b_{\beta 3}\varepsilon_3 + b_{\beta 4}\varphi_1 + b_{\beta 5}\varphi_2 + b_{\beta 6}\varphi_3$$

(21)

then the 18 coefficients $b_{\beta i}$ ($\beta = 1, 2, 3; i = 1, 2, \ldots, 6$) form a matrix that represents, in abridged notation, a fourth-order tensor $b_{\alpha\gamma\lambda\mu}$ that is antisymmetric in α and γ (because the axial vector $\mathbf{G}$ is a second-order antisymmetric tensor) and symmetric in λ and μ (because the pure deformations constitute a second-order symmetric tensor). β stands for the pair $\alpha\gamma$, while i stands for the pair $\lambda\mu$.

With the help of equations (20) and (21) and the elastic matrices given in Appendix 1, it is possible to write the components of the matrices $b_{\beta i}$ in terms of the elastic constants c_{ij} for each of the Laue-groups. These matrices are given in Appendix 2.

It is interesting to notice that for each Laue-group the number of independent components of the matrices $b_{\beta i}$ is equal to the difference between the number of independent elastic constants c_{ij} in the theory of Voigt and the number proposed in this paper. Besides,

these matrices show for each Laue-group the nature of the difference between the two schemes.

5. The strain-energy function

An assumption regarding the body-couples was made in section 3. This assumption states that body-couples can arise only at points of the crystal at which there is a deformation. As a consequence of this assumption, the six relations (17) were obtained, which reduced the number of independent elastic constants from 45 to 39. If these relations (17) are introduced into the strain-energy function

$$W = \tfrac{1}{2} \sum_{ij} c_{ij} \varepsilon_i \varepsilon_j \quad (i, j = 1, 2, \ldots, 9), \quad (22)$$

some further interesting results are obtained.

First, the substitutions (13) are introduced into equation (22), so that the strain energy becomes a function of the nine parameters ε_α, φ_α and θ_α ($\alpha = 1, 2, 3$). The function W can now be written as a sum of three terms:

$$W = W_d + W_{dr} + W_r \quad (23)$$

in such a way that W_d contains deformations only (terms in $\varepsilon_\alpha \varepsilon_\beta$, $\varepsilon_\alpha \varphi_\beta$ and $\varphi_\alpha \varphi_\beta$); W_{dr} contains both deformations and rotations (terms in $\varepsilon_\alpha \theta_\beta$ and $\varphi_\alpha \theta_\beta$); and W_r contains rotations only (terms in $\theta_\alpha \theta_\beta$). The subscripts d and r refer respectively to *deformations* and *rotations*, while the subscript dr indicates that there are both deformations and rotations.

The expressions of the three terms in equation (23) are (all summations over α and β, from 1 to 3):

$$W_d = \tfrac{1}{2} \Sigma c_{\alpha\beta} \varepsilon_\alpha \varepsilon_\beta + \Sigma (c_{\alpha, \beta+3} + c_{\alpha, \beta+6}) \varepsilon_\alpha \varphi_\beta$$
$$+ \tfrac{1}{2} \Sigma (c_{\alpha+3, \beta+3} + c_{\alpha+6, \beta+6} + 2c_{\alpha+3, \beta+6}) \varphi_\alpha \varphi_\beta$$
$$W_{dr} = \Sigma (c_{\alpha, \beta+6} - c_{\alpha, \beta+3}) \varepsilon_\alpha \theta_\beta$$
$$+ \Sigma (c_{\alpha+6, \beta+6} + c_{\alpha+3, \beta+6} - c_{\alpha+6, \beta+3} - c_{\alpha+3, \beta+3}) \varphi_\alpha \theta_\beta$$
$$W_r = \tfrac{1}{2} \Sigma (c_{\alpha+3, \beta+3} + c_{\alpha+6, \beta+6} - c_{\alpha+3, \beta+6} - c_{\alpha+6, \beta+3}) \theta_\alpha \theta_\beta .$$
$$(24)$$

It can be seen that, if use is made of equation (16), it follows that $W_r = 0$, because of the vanishing of the coefficients of $\theta_\alpha \theta_\beta$. Therefore, from the assumption made in section 3 regarding the body-couples, it follows as a necessary consequence that equation (23) should be reduced to:

$$W = W_d + W_{dr} . \quad (25)$$

In physical terms, this means that the rotational part of the strain can also contribute to the strain energy if there is a deformation present ($W_{dr} \neq 0$); but the rotations do not contribute to the strain-energy if they are not accompanied by a deformation ($W_r = 0$). This result can be summed up in the following way.

The assumption has been made that:

(i) Strain-induced body-couples cannot arise at doints of zero deformation.

From this assumption it follows, as has been shown above, that:

(ii) The strain energy is invariant under rotations at points of zero deformation.

It is interesting to notice that the argument that leads from (i) to (ii) can also be followed the other way round: if (ii) is used as a starting assumption, then (i) and the six relations (17) follow as a necessary consequence.

It is sometimes convenient to express these ideas in terms of 'elements of volume'. These elements of volume are small compared with the wavelength of the elastic waves, so that the strains and the stresses can be considered as constant throughout each element of volume. They are, on the other hand, large compared with the size of the unit cell, which makes it possible to apply to crystal lattices notions that belong to the continuum. In other words, 'element of volume' is to be understood as a portion of a lattice throughout which the lattice strain is constant. The two statements, (i) and (ii), can then also be formulated as follows:

(i) Strain-induced body-couples cannot arise in undeformed elements of volume.

(ii) Rotations of undeformed elements of volume leave the strain energy invariant.

The inclusion in the expression for the strain energy of terms that depend on rotations (equations (23) and (24)) requires some further comment. Whatever model is chosen to explain the strain-induced body-couples, these rotations can contribute to the strain energy only if they are rotations of part of the matter relative to the surrounding matter in the same crystal. Such a relative rotation is absent in rigid rotations so that, as should be expected, rigid rotations do not contribute to the rotational part of the strain energy.

The separation of W into its three terms W_d, W_{dr} and W_r (equations (23) and (24)) enables one to interpret the different assumptions made in the various theories of crystal elasticity.

(i) In the Voigt theory, and also in the lattice theory of Born & Huang (1954), the strain energy is invariant under rotations of any element of volume, whether it is deformed or not; the rotations do not come into the strain-energy function. This amounts to keeping the first term, W_d, only:

$$W = W_d . \quad (26)$$

But this also carries with it an *a priori* denial of the possible existence of strain-induced body-couples.

(ii) In the theory of Laval and LeCorre, on the other hand, no invariance under rotations is specified at all for the strain energy; all rotations are considered,

whether there is a deformation or not. In this way, all three terms of W are kept:

$$W = W_d + W_{dr} + W_r. \tag{23}$$

(iii) In the present work, the strain energy is taken as invariant under rotations, but this invariance is stated *a priori* only for undeformed elements of volume. The rotations have to be considered, but they can only contribute to the strain energy if in that particular element of volume there is a deformation present at the same time. This means keeping the two terms W_d and W_{dr} while W_r vanishes:

$$W = W_d + W_{dr}. \tag{25}$$

Finally, may it be added that a clear distinction should be made between the following two cases:

(*a*) The rotations θ are not a function of $\mathbf{x}$, that is, they are constant throughout the part of the crystal under study; in this case the rotations are rigid rotations, and at no point is there a rotation relative to the surrounding matter. Consequently, $W_{dr} = 0$.

(*b*) The rotations θ are a function of $\mathbf{x}$ and t (time); they constitute the rotations that accompany the deformations in every element of volume in cases such as the propagation of stress waves in crystals. These rotations contribute to the strain energy through the term W_{dr}. But, if a rigid rotation is superimposed on these rotations no further contribution is made towards the strain energy.

Another way of looking at the distinction made between the two cases (*a*) and (*b*) is this: in dynamic elasticity, once an arbitrary system of reference has been fixed relative to the unstrained crystal, any other system obtained by means of a rigid rotation of the former can also be used. But rotations of the reference system that are a function of $\mathbf{x}$ and t must be excluded. This point is mentioned here because in dynamic elasticity the rotation θ due to the strain is not only a function of $\mathbf{x}$ but at every point it is also a function of t; and the neglecting of the term W_{dr} (through the neglecting of the rotation θ) amounts to the unacceptable assumption that at every point the axes of reference may librate.

6. On the possibility of obtaining experimental evidence

In order to decide which of the three schemes of elastic constants applies to a given crystal some appropriate elastic constants have to be determined experimentally. Dynamic methods, such as Bergmann–Schaefer elastograms or pulse-velocity measurements, are the most adequate; they can decide, for most crystals, whether the scheme of Voigt does apply or not. The propagation of elastic waves in crystals, which is the foundation of all dynamic methods, has been comprehensively reviewed by Musgrave

(1959). The test is whether certain elastic constants —which are equal in the Voigt theory but not necessarily so in the Laval theory or in the present formulation—are in fact equal or not.

Zubov & Firsova (1956) found for quartz a difference of just over 1% between c_{44} and c_{55} and of 3% between c_{14} and c_{17}; these differences should be zero if the scheme of Voigt were applicable.

The case of ammonium dihydrogen phosphate (ADP) will be considered more in detail. For this crystal it is a possible difference between c_{44} and c_{55} that is relevant. New measurements of some of the elastic constants of ADP have been made (Joel & Wooster, 1960) by means of Bergmann–Schaefer elastograms— a method particularly suited to give good relative values of the elastic constants—and a difference of 6%($\pm$2%) was found. That is, the experimental results cannot be fitted into the matrix of Voigt (6 independent components) because of $c_{44} \neq c_{55}$.

The experimental results for ADP can, however, be interpreted satisfactorily both with the matrix of Laval–LeCorre (9 independent components) and with the one suggested now (7 independent components). In the matrix of Laval–LeCorre c_{47} and c_{69} are independent constants while in the matrix shown in Appendix 1 (Laue-group $4/mmm$) these elastic constants have to satisfy the relations $2c_{47} = c_{44} + c_{55}$ and $c_{69} = c_{66}$, which are the relations (17) for ADP. However, dynamical experiments can only give for ADP the seven values of: $c_{11}, c_{33}, c_{44}, c_{55}, c_{66}, c_{12} + c_{69}$ and $c_{13} + c_{47}$. Therefore, the evidence as to whether c_{47} and c_{69} are independent constants or not, that is, the adoption for ADP of either of the two schemes of elastic constants (the Laval–LeCorre one or the present one) cannot be decided by dynamical experiments alone.

In more general terms, the situation can be described thus: the expressions that give the velocity of propagation of elastic waves in crystals, expressions on which the dynamic methods are based, contain the elastic constants in sums such as $c_{\alpha\beta\gamma\delta} + c_{\alpha\delta\gamma\beta}$. When $\alpha = \gamma$ and/or $\beta = \delta$, each of these sums reduces to one single elastic constant (there are 27 such cases). But when both $\alpha \neq \gamma$ and $\beta \neq \delta$, the two elastic constants $c_{\alpha\beta\gamma\delta}$ and $c_{\alpha\delta\gamma\beta}$ cannot be measured separately. It is this dynamic coupling that makes it impossible to decide, by means of dynamical experiments, whether the six relations

$$c_{\alpha\beta\gamma\delta} + c_{\beta\alpha\delta\gamma} = c_{\alpha\beta\delta\gamma} + c_{\beta\alpha\gamma\delta} \tag{18}$$

are in fact satisfied, that is, whether the elastic matrices of Laval–LeCorre or the present ones give a more accurate description of the elastic properties of crystals. As only static measurements can separate the elastic constants that appear coupled in the dynamical equations, this question will remain a theoretical one until static measurements on suitable crystals, of sufficient accuracy to allow them to be related to the dynamic results, become available.

APPENDIX 1

The new elastic matrices for the 11 Laue-groups and for isotropic solids

For each of the crystal classes the symmetry imposes restrictions on the number of independent elastic constants. The procedure for determining these can be consulted in books such as those of Wooster (1949) and Nye (1957) where also the classical elastic matrices can be consulted. In tabulating the results it is worthwhile noticing that, as the presence or absence of a centre of symmetry does not affect the fourth-order elastic tensor $c_{\alpha\beta\gamma\delta}$, the number of different cases in the 32 crystal classes is actually 11, the number of Laue-groups. The new matrices are obtained simply by applying the conditions (17) to the elastic matrices of LeCorre (1953).

The new elastic matrices (in the abridged notation) for the 11 Laue-groups, and also the one corresponding to isotropic (amorphous) solids, are given below. In these matrices, whenever symmetry requirements or any of the conditions (17) lead to the vanishing of a given constant or to the equalization of two constants, with the same or with opposite sign, this will be introduced in the matrix. The vanishing components are indicated by a dot. The relations involving several constants are given below each matrix; also the number of independent components is indicated.

$$1, \bar{1}$$

```
11  12  13  14  15  16  17  18  19
12  22  23  24  25  26  27  28  29
13  23  33  34  35  36  37  38  39
14  24  34  44  45  46  47  48  49
15  25  35  45  55  56  57  58  59
16  26  36  46  56  66  67  68  69
17  27  37  47  57  67  77  78  79
18  28  38  48  58  68  78  88  89
19  29  39  49  59  69  79  89  99
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$$47 = \tfrac{1}{2}(44+77)$$
$$58 = \tfrac{1}{2}(55+88)$$
$$69 = \tfrac{1}{2}(66+99)$$
$$45+78 = 48+57$$
$$56+89 = 59+68$$
$$46+79 = 49+67$$

(39 independent components)

$$2, m, 2/m$$

```
11  12  13   .  15   .   .  18   .
12  22  23   .  25   .   .  28   .
13  23  33   .  35   .   .  38   .
 .   .   .  44   .  46  47   .  49
15  25  35   .  55   .   .  58   .
 .   .   .  46   .  66  67   .  69
 .   .   .  47   .  67  77   .  79
18  28  38   .  58   .   .  88   .
 .   .   .  49   .  69  79   .  99
```

$$47 = \tfrac{1}{2}(44+77)$$
$$58 = \tfrac{1}{2}(55+88)$$
$$69 = \tfrac{1}{2}(66+99)$$
$$46+79 = 49+67$$

(21 independent components)

$$222, mm2, mmm$$

```
11  12  13   .   .   .   .   .   .
12  22  23   .   .   .   .   .   .
13  23  33   .   .   .   .   .   .
 .   .   .  44   .   .  47   .   .
 .   .   .   .  55   .   .  58   .
 .   .   .   .   .  66   .   .  69
 .   .   .  47   .   .  77   .   .
 .   .   .   .  58   .   .  88   .
 .   .   .   .   .  69   .   .  99
```

$$47 = \tfrac{1}{2}(44+77)$$
$$58 = \tfrac{1}{2}(55+88)$$
$$69 = \tfrac{1}{2}(66+99)$$

(12 independent components)

$$3, \bar{3}$$

```
 11   12   13   14   15   16   17   18  -16
 12   11   13  -14  -15   16  -17  -18  -16
 13   13   33    .    .   36    .    .  -36
 14  -14    .   44   45  -18   47    .  -18
 15  -15    .   45   55   17    .   47   17
 16   16   36  -18   17   66  -15   14   66
 17  -17    .   47    .  -15   55  -45  -15
 18  -18    .    .   47   14  -45   44   14
-16  -16  -36  -18   17   66  -15   14   66
```

$$47 = \tfrac{1}{2}(44+55)$$
$$66 = \tfrac{1}{2}(11-12)$$

(13 independent components)

$$32, 3m, \bar{3}m$$

```
11  12  13  14   .   .  17   .   .
12  11  13 -14   .   . -17   .   .
13  13  33   .   .   .   .   .   .
14 -14   .  44   .   .  47   .   .
 .   .   .   .  55  17   .  47  17
 .   .   .   .  17  66   .  14  66
17 -17   .  47   .   .  55   .   .
 .   .   .   .  47  14   .  44  14
 .   .   .   .  17  66   .  14  66
```

$$47 = \tfrac{1}{2}(44+55)$$
$$66 = \tfrac{1}{2}(11-12)$$

(8 independent components)

$$4, \bar{4}, 4/m$$

```
 11   12   13    .    .  16   .   . -26
 12   11   13    .    .  26   .   . -16
 13   13   33    .    .  36   .   . -36
  .    .    .   44   45   .  47   .   .
  .    .    .   45   55   .   .  47   .
 16   26   36    .    .  66   .   .  66
  .    .    .   47    .   .  55 -45   .
  .    .    .    .   47   . -45  44   .
-26  -16  -36    .    .  66   .   .  66
```

$$47 = \tfrac{1}{2}(44+55)$$

(11 independent components)

422, 4̄2m, 4mm, 4/mmm

```
11  12  13   .    .    .    .    .    .
12  11  13   .    .    .    .    .    .
13  13  33   .    .    .    .    .    .
 .   .   .  44    .    .   47    .    .
 .   .   .   .   55    .    .   47    .
 .   .   .   .    .   66    .    .   66
 .   .   .  47    .    .   55    .    .
 .   .   .   .   47    .    .   44    .
 .   .   .   .    .   66    .    .   66
```

$47 = \frac{1}{2}(44+55)$

(7 independent components)

6, 6̄, 6/m

```
 11   12   13    .    .   16    .    .  -16
 12   11   13    .    .   16    .    .  -16
 13   13   33    .    .   36    .    .  -36
  .    .    .   44   45    .   47    .    .
  .    .    .   45   55    .    .   47    .
 16   16   36    .    .   66    .    .   66
  .    .    .   47    .    .   55  -45    .
  .    .    .    .   47    .  -45   44    .
-16  -16  -36    .    .   66    .    .   66
```

$47 = \frac{1}{2}(44+55)$
$66 = \frac{1}{2}(11-12)$

(9 independent components)

622, 6̄m2, 6mm, 6/mmm

```
11  12  13   .    .    .    .    .    .
12  11  13   .    .    .    .    .    .
13  13  33   .    .    .    .    .    .
 .   .   .  44    .    .   47    .    .
 .   .   .   .   55    .    .   47    .
 .   .   .   .    .   66    .    .   66
 .   .   .  47    .    .   55    .    .
 .   .   .   .   47    .    .   44    .
 .   .   .   .    .   66    .    .   66
```

$47 = \frac{1}{2}(44+55)$
$66 = \frac{1}{2}(11-12)$

(6 independent components)

23, m3

```
11  12  12   .    .    .    .    .    .
12  11  12   .    .    .    .    .    .
12  12  11   .    .    .    .    .    .
 .   .   .  44    .    .   47    .    .
 .   .   .   .   44    .    .   47    .
 .   .   .   .    .   44    .    .   47
 .   .   .  47    .    .   77    .    .
 .   .   .   .   47    .    .   77    .
 .   .   .   .    .   47    .    .   77
```

$47 = \frac{1}{2}(44+77)$

(4 independent components)

432, 4̄3m, m3m

```
11  12  12   .    .    .    .    .    .
12  11  12   .    .    .    .    .    .
12  12  11   .    .    .    .    .    .
 .   .   .  44    .    .   44    .    .
 .   .   .   .   44    .    .   44    .
 .   .   .   .    .   44    .    .   44
 .   .   .  44    .    .   44    .    .
 .   .   .   .   44    .    .   44    .
 .   .   .   .    .   44    .    .   44
```

(3 independent components)

Isotropic solids

```
11  12  12   .    .    .    .    .    .
12  11  12   .    .    .    .    .    .
12  12  11   .    .    .    .    .    .
 .   .   .  44    .    .   44    .    .
 .   .   .   .   44    .    .   44    .
 .   .   .   .    .   44    .    .   44
 .   .   .  44    .    .   44    .    .
 .   .   .   .   44    .    .   44    .
 .   .   .   .    .   44    .    .   44
```

$44 = \frac{1}{2}(11-12)$

(2 independent components)

APPENDIX 2

The body-couples as functions of the deformations

Equations (20) and (21), section 4, define the matrix $b_{\beta i}$ ($\beta = 1, 2, 3$; $i = 1, 2, \ldots, 6$) that gives the moments per unit volume exerted by the body-couples in terms of the deformations:

	ε_1	ε_2	ε_3	φ_1	φ_2	φ_3
G_1	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
G_2	b_{21}	b_{22}	b_{23}	b_{24}	b_{25}	b_{26}
G_3	b_{31}	b_{32}	b_{33}	b_{34}	b_{35}	b_{36}

The components of the matrices $b_{\beta i}$ can be written in terms of the elastic constants for each of the Laue-groups and for isotropic solids; they are given below. Only the subscripts are written, so that $2(45-48)$ means $2(c_{45} - c_{48})$, $2(36)$ means $2c_{36}$, etc.

1, 1̄ (18 independent components)

```
14-17   24-27   34-37    44-77     2(54-57)  2(64-67)
15-18   25-28   35-38   2(45-48)    55-88    2(65-68)
16-19   26-29   36-39   2(46-49)   2(56-59)   66-99
```

2, m, 2/m (8 independent components)

```
  .       .       .      44-77       .       2(64-67)
15-18   25-28   35-38     .        55-88        .
  .       .       .     2(46-49)     .        66-99
```

222, mm2, mmm (3 independent components)

```
  .   .   .   44-77     .         .
  .   .   .     .      55-88       .
  .   .   .     .        .       66-99
```

3, $\bar{3}$ (6 independent components)

14−17	17−14	.	44−55	2(45)	2(15−18)
15−18	18−15	.	2(45)	55−44	2(17−14)
2(16)	2(16)	2(36)	.	.	.

32, 3m, $\bar{3}$m   (2 independent components

14−17	17−14	.	44−55	.	.
.	.	.	.	55−44	2(17−14)
.	.	.	.	.	.

4, $\bar{4}$, 4/m   (4 independent components)

.	.	.	44−55	2(45)	.
.	.	.	2(45)	55−44	.
16+26	16+26	2(36)	.	.	.

422, $\bar{4}$2m, 4mm, 4/mmm   (1 independent component)

.	.	.	44−55	.	.
.	.	.	.	55−44	.
.	.	.	.	.	.

6, $\bar{6}$, 6/m   (4 independent components)

.	.	.	44−55	2(45)	.
.	.	.	2(45)	55−44	.
2(16)	2(16)	2(36)	.	.	.

622, $\bar{6}$m2, 6mm, 6/mmm   (1 independent component)

.	.	.	44−55	.	.
.	.	.	.	55−44	.
.	.	.	.	.	.

23, m3   (1 independent component)

.	.	.	44−77	.	.
.	.	.	.	44−77	.
.	.	.	.	.	44−77

432, $\bar{4}$3m, m3m, and isotropic solids   (all components vanish)

.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

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Resumen

En este trabajo se sugiere que en aquellos casos en los cuales es necesario usar el sistema de 45 constantes elásticas de Laval hay seis relaciones que deben ser satisfechas por algunas de las constantes elásticas. Estas relaciones resultan de un análisis de los torques de volumen y de la energía de deformación. Como consecuencia, el número de constantes elásticas independientes de un cristal triclínico se reduce de 45 a 39, con las reducciones correspondientes en las demás clases cristalinas. Las nuevas matrices elásticas para cada uno de los once grupos de Laue y para sólidos isótropos están indicadas en un apéndice al final de este trabajo. También se da una discusión de las diversas hipótesis hechas por diferentes autores en relación con la invariancia rotacional de la energía de deformación.

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